Some Aspects of Barua's Work on Automata Theory

Palash Sarkar

Applied Statistics Unit
Indian Statistical Institute

As part of the function to celebrate
the 60th birthday
of
Professor Rana Barua

16th November 2012.

Extent of Rana-da's Interest

- Descriptive set theory.
- Automata on infinite strings.
- Cellular automata.
- Algorithms and VLSI architectures for (field) arithmetic.
- Circuit complexity, DNA computing.
- Voting games.
- Cryptography.

Extent of Rana-da's Interest

- Descriptive set theory.
- Automata on infinite strings.
- Cellular automata.
- Algorithms and VLSI architectures for (field) arithmetic.
- Circuit complexity, DNA computing.
- Voting games.
- Cryptography.

This talk will highlight.

- Barua hierarchy.
- Merlin's game and σ- (σ^+)-automata.

Barua Hierarchy

Automata on Infinite Strings

Consider an FA $\mathcal{M}$ running on a one-way infinite string α.

Let $\mathrm{Inf}(\alpha)$ be the set of states that are visited infinitely often.

Acceptance condition: α is accepted by $\mathcal{M}$ if

- Büchi: $\mathrm{Inf}(\alpha)$ contains a final state.
- Muller: $\mathrm{Inf}(\alpha)$ is a set of final states.

Automata on Infinite Strings

Consider an FA $\mathcal{M}$ running on a one-way infinite string α.

Let $\mathrm{Inf}(\alpha)$ be the set of states that are visited infinitely often.

Acceptance condition: α is accepted by $\mathcal{M}$ if

- Büchi: $\mathrm{Inf}(\alpha)$ contains a final state.
- Muller: $\mathrm{Inf}(\alpha)$ is a set of final states.

A collection L of one-way infinite strings is a

- Deterministic ω-regular language: if $L = L(\mathcal{M})$ for some deterministic Büchi automaton $\mathcal{M}$;
- ω-regular language: if $L = L(\mathcal{M})$ for some non-deterministic Büchi automaton $\mathcal{M}$.

Automata on Infinite Strings

Consider an FA $\mathcal{M}$ running on a one-way infinite string α.

Let $\mathrm{Inf}(\alpha)$ be the set of states that are visited infinitely often.

Acceptance condition: α is accepted by $\mathcal{M}$ if

- Büchi: $\mathrm{Inf}(\alpha)$ contains a final state.
- Muller: $\mathrm{Inf}(\alpha)$ is a set of final states.

A collection L of one-way infinite strings is a

- Deterministic ω-regular language: if $L = L(\mathcal{M})$ for some deterministic Büchi automaton $\mathcal{M}$;
- ω-regular language: if $L = L(\mathcal{M})$ for some non-deterministic Büchi automaton $\mathcal{M}$.

Theorem (Büchi-McNaughton): The class $\mathcal{R}$ of all ω-regular languages is the Boolean closure of all deterministic ω-regular languages.

Theorem (McNaughton): L is ω-regular iff $L = L(\mathcal{M})$ for some deterministic Muller automaton $\mathcal{M}$.

Barua Hierarchy: The Approach

An ω-language L is said to be rational if there is a decreasing sequence of *regular* languages $X_0 \supseteq X_1 \supseteq \cdots$ such that

- $X_i \setminus X_{i+1}$ is finite;
- $\bigcap_{i \geq 0} X_i = \emptyset$;
- $L = \bigcap_{i \geq 0} X_i \Sigma^\omega$.

Proposition: An ω-language is rational iff it is deterministic ω-regular.

Barua Hierarchy: The Approach

Theorem: L is ω-regular iff there is a decreasing sequence $G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n$ of rational sets such that

$$L = \bigcup_{i \text{ even}} (G_i \setminus G_{i+1}).$$

Barua Hierarchy: The Approach

Theorem: L is ω-regular iff there is a decreasing sequence $G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n$ of rational sets such that

$$L = \bigcup_{i \text{ even}} (G_i \setminus G_{i+1}).$$

For each $n \geq 0$, define $\mathcal{R}_{n+1}$ to be the set of all L which is the union of differences of a sequence of rational sets of length n.

Barua Hierarchy: The Approach

Theorem: L is ω-regular iff there is a decreasing sequence $G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n$ of rational sets such that

$$L = \bigcup_{i \text{ even}} (G_i \setminus G_{i+1}).$$

For each $n \geq 0$, define $\mathcal{R}_{n+1}$ to be the set of all L which is the union of differences of a sequence of rational sets of length n.

(Question: Is it possible for L to be "realised" by two sequences of rational sets of *different* lengths?)

Barua Hierarchy: The Approach

Theorem: L is ω-regular iff there is a decreasing sequence $G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n$ of rational sets such that

$$L = \bigcup_{i \text{ even}} (G_i \setminus G_{i+1}).$$

For each $n \geq 0$, define $\mathcal{R}_{n+1}$ to be the set of all L which is the union of differences of a sequence of rational sets of length n.

(Question: Is it possible for L to be "realised" by two sequences of rational sets of *different* lengths?) If so:

- The "length" of L should be the *minimum* of the lengths of all such sequences.
- $\mathcal{R}_{n+1}$ should consist of all L of "length" *at most* n.

The Barua Hierarchy (1992)

$$\mathcal{R}_1 \subseteq \mathcal{R}_2 \subseteq \mathcal{R}_3 \subseteq \cdots ; \text{ and } \mathcal{R} = \bigcup_{n \geq 1} \mathcal{R}_n.$$

The Barua Hierarchy (1992)

$$\mathcal{R}_1 \subseteq \mathcal{R}_2 \subseteq \mathcal{R}_3 \subseteq \cdots ; \text{ and } \mathcal{R} = \bigcup_{n \geq 1} \mathcal{R}_n.$$

Theorem: The hierarchy does not collapse.

Topological Considerations

- Endow Σ^ω with the product of discrete topologies.
- Σ^ω is a compact metric space with respect to the following metric: $d(\alpha, \alpha) = 0$ and for $\alpha \neq \beta$, $d(\alpha, \beta) = 1/2^i$ where i is the *first* position where the two strings differ.

Topological Considerations

- Endow Σ^ω with the product of discrete topologies.
- Σ^ω is a compact metric space with respect to the following metric: $d(\alpha, \alpha) = 0$ and for $\alpha \neq \beta$, $d(\alpha, \beta) = 1/2^i$ where i is the *first* position where the two strings differ.

Some Standard Notions:

- *Complete metric space:* every Cauchy sequence converges to a limit in the metric space.
- *Separable space:* there is at least one countable dense set, i.e., a sequence $\{x_n\}$ such that any non-empty open set contains at least one point from the sequence.

Topological Considerations

- Endow Σ^ω with the product of discrete topologies.
- Σ^ω is a compact metric space with respect to the following metric: $d(\alpha, \alpha) = 0$ and for $\alpha \neq \beta$, $d(\alpha, \beta) = 1/2^i$ where i is the *first* position where the two strings differ.

Some Standard Notions:

- *Complete metric space:* every Cauchy sequence converges to a limit in the metric space.
- *Separable space:* there is at least one countable dense set, i.e., a sequence $\{x_n\}$ such that any non-empty open set contains at least one point from the sequence.
- G_δ *set:* countable intersection of open sets;
  F_σ set: countable union of closed sets.
- $G_{\delta\sigma}$ *set:* countable union of G_δ sets;
  $F_{\sigma\delta}$ *set:* countable intersection of F_σ sets;

Hausdorff-Kuratowski Hierarchy and Barua Hierarchy

Theorem (Kuratowski): A set A in a complete separable metric space is both $F_{\sigma\delta}$ and $G_{\delta\sigma}$ iff there is a decreasing sequence $\{G_\lambda : \lambda \leq n\}$ of G_δ sets such that

$$A = \bigcup_{\lambda \text{ even}} (G_\lambda \setminus G_{\lambda+1}).$$

Hausdorff-Kuratowski Hierarchy and Barua Hierarchy

Theorem (Kuratowski): A set A in a complete separable metric space is both $F_{\sigma\delta}$ and $G_{\delta\sigma}$ iff there is a decreasing sequence $\{G_\lambda : \lambda \leq n\}$ of G_δ sets such that

$$A = \bigcup_{\lambda \text{ even}} (G_\lambda \setminus G_{\lambda+1}).$$

HK-Difference hierarchy: denote the nth level by $\mathcal{D}_n$.

Theorem: $\mathcal{R} \subseteq \bigcup_{n \geq 1} \mathcal{D}_n$.

Hausdorff-Kuratowski Hierarchy and Barua Hierarchy

Theorem (Kuratowski): A set A in a complete separable metric space is both $F_{\sigma\delta}$ and $G_{\delta\sigma}$ iff there is a decreasing sequence $\{G_\lambda : \lambda \leq n\}$ of G_δ sets such that

$$A = \bigcup_{\lambda \text{ even}} (G_\lambda \setminus G_{\lambda+1}).$$

HK-Difference hierarchy: denote the nth level by $\mathcal{D}_n$.

Theorem: $\mathcal{R} \subseteq \bigcup_{n \geq 1} \mathcal{D}_n$.

Define: $\overline{\mathcal{D}}_n = \mathcal{D}_n \cap \mathcal{R}$.

Hausdorff-Kuratowski Hierarchy and Barua Hierarchy

Theorem (Kuratowski): A set A in a complete separable metric space is both $F_{\sigma\delta}$ and $G_{\delta\sigma}$ iff there is a decreasing sequence $\{G_\lambda : \lambda \leq n\}$ of G_δ sets such that

$$A = \bigcup_{\lambda \text{ even}} (G_\lambda \setminus G_{\lambda+1}).$$

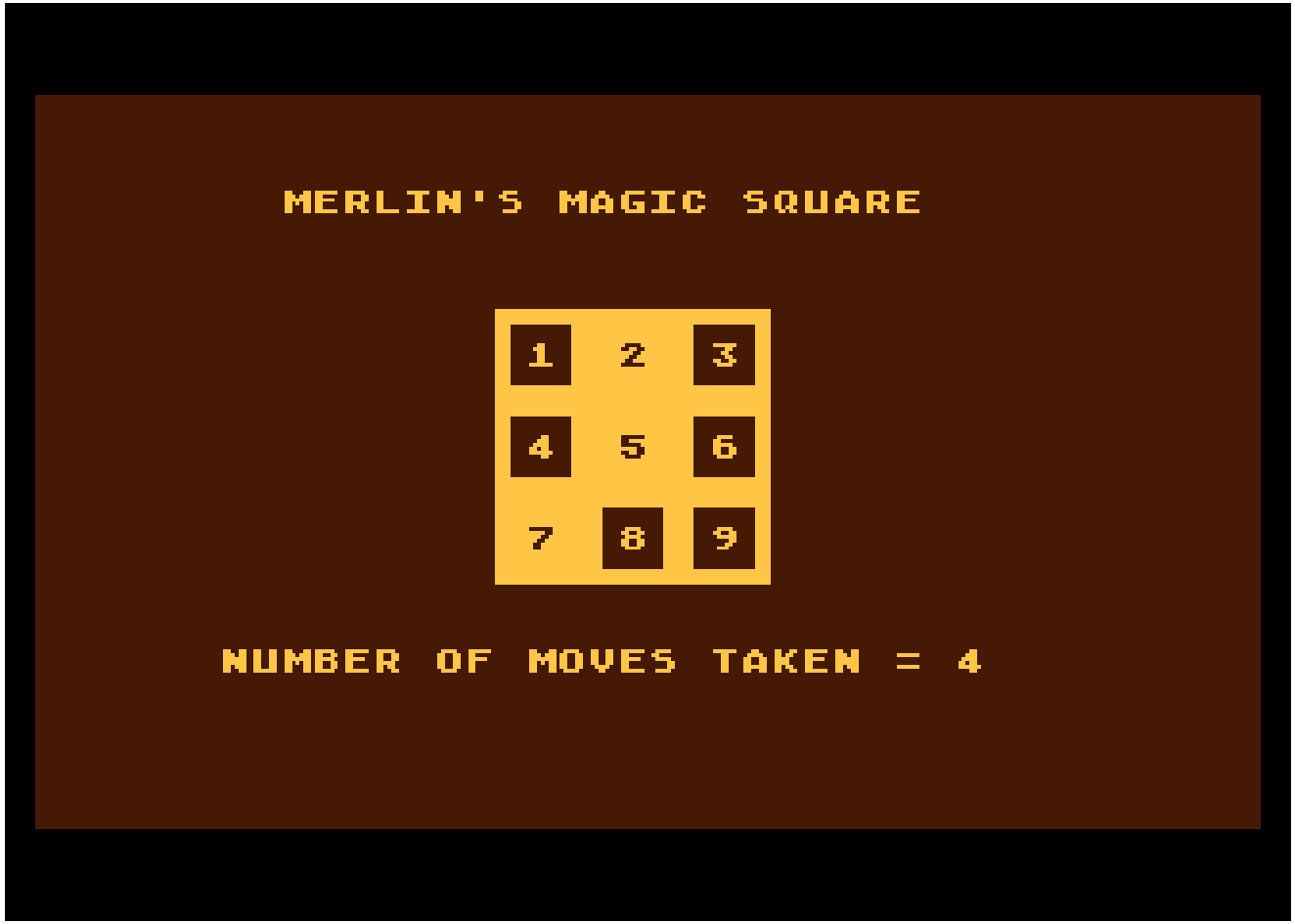
HK-Difference hierarchy: denote the nth level by $\mathcal{D}_n$.

Theorem: $\mathcal{R} \subseteq \bigcup_{n \geq 1} \mathcal{D}_n$.

Define: $\overline{\mathcal{D}}_n = \mathcal{D}_n \cap \mathcal{R}$.

Theorem (Barua): $\mathcal{R}_n = \overline{\mathcal{D}}_n$.

Merlin's Magic Square and σ-Automata



From `atarimania.com`

Merlin's Magic Square: Rules

$n \times n$ square: each cell has two states – ON or OFF.

- First player: chooses a start and end configuration.
- Second player: makes a sequence of moves which lead from the start to the end.
- A move is to choose a cell. Choosing a cell results in changing the states of *all* neighbouring cells.

Merlin's Magic Square: Rules

$n \times n$ square: each cell has two states – ON or OFF.

- First player: chooses a start and end configuration.
- Second player: makes a sequence of moves which lead from the start to the end.
- A move is to choose a cell. Choosing a cell results in changing the states of *all* neighbouring cells.

σ-Game (introduced by Sutner):

- Two person game played on a graph.
- Each vertex has two states – 0 or 1.
- Roles of first and second player as also the rule for a move are the same as above.

Merlin's Magic Square: Rules

$n \times n$ square: each cell has two states – ON or OFF.

- First player: chooses a start and end configuration.
- Second player: makes a sequence of moves which lead from the start to the end.
- A move is to choose a cell. Choosing a cell results in changing the states of *all* neighbouring cells.

σ-Game (introduced by Sutner):

- Two person game played on a graph.
- Each vertex has two states – 0 or 1.
- Roles of first and second player as also the rule for a move are the same as above.

σ^+-game: The state of the chosen vertex also changes.

Both games are special cases of cellular automata

Modelling the σ-(σ^+-)Game: Sutner (1990)

- The order of moves, i.e., the order in which the vertices are chosen does not matter.
- A solution is a *set* of vertices and hence can be thought of as a configuration.

Modelling the σ-(σ^+-)Game: Sutner (1990)

- The order of moves, i.e., the order in which the vertices are chosen does not matter.
- A solution is a *set* of vertices and hence can be thought of as a configuration.
- Let $\sigma(Z)$ be the configuration obtained from **0** when the configuration Z is chosen by the second player. Similarly define $\sigma^+(Z)$.
- $\sigma(Z)$ and $\sigma^+(Z)$ are linear maps.
- Number of solutions for a pair (X_s, X_t) is 2^t where t is the dimension of the kernel of the linear map.

Modelling the σ-(σ^+-)Game: Sutner (1990)

- The order of moves, i.e., the order in which the vertices are chosen does not matter.
- A solution is a *set* of vertices and hence can be thought of as a configuration.
- Let $\sigma(Z)$ be the configuration obtained from **0** when the configuration Z is chosen by the second player. Similarly define $\sigma^+(Z)$.
- $\sigma(Z)$ and $\sigma^+(Z)$ are linear maps.
- Number of solutions for a pair (X_s, X_t) is 2^t where t is the dimension of the kernel of the linear map.
- The number of solutions for the σ-game on an $n \times m$ grid is 2^k where $k = \gcd(m+1, n+1) - 1$.

Barua-Ramakrishnan (1996)

Study of σ-(σ^+-)game on 2-dimensional grid, cylinders and tori.

- Algebraic analysis using a sequence of polynomials over $GF(2)$.
- An algorithm for finding the dimension of the kernel of σ^+.
- Characterisation of invertibility of the σ^+ map.
- Count of the number of solutions of the σ^+-game in special cases.

Sarkar-Barua (1998)

Study of σ-(σ^+-)game on higher-dimensional grid, cylinders and tori.

- A matrix representation of the map as a sum of Kronecker products of certain patterned matrices.
- A recursive expression for the characteristic polynomial of the matrix in terms of resultants and binary Chebyshev polynomials.
- General results on the invertibility of the corresponding automata.
- Relation of invertibility to number-theoretic properties of the number of dimensions and the lengths of the grids.

Concluding Remark

Professor Rana Barua is interested in any topic that has a mathematical content.

Concluding Remark

Professor Rana Barua is interested in any topic that has a mathematical content.

Thank you for your attention!