A Brief History of Scientific Thoughts
Lecture 2

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Geometry From Observations

Suppose one draws 5 triangles $T_1, \ldots, T_5$ on the plane and measures each angle of each triangle.
Let a_i, b_i and c_i be the measures of the angles of T_i.
Observation:

$$a_1 + b_1 + c_1 = \cdots = a_5 + b_5 + c_5 = 180^\circ.$$

Questions:

- Can one conclude that the sum three angles of any triangle is 180°?
- If 5 is changed to one million, then can one conclude the above?

Similar other questions can be asked about geometry?



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Number Theory From Observations

Let $f_n = 2^{2^n} + 1$.

- $f_0 = 3, f_1 = 5, f_2 = 17, f_3 = 257, f_4 = 65537$ are primes.
- Question: are all f_n prime? (due to Fermat)
- $f_5 = 4294967297 = 641 \times 6700417$ (due to Euler).

3 is between 2 and 4;
5 is between 3 and 6;
7 is between 4 and 8;
7 is between 5 and 10;
11 is between 6 and 12;

- Question: is there always a prime between a number and its double?

Bertrand-Chebyshev Theorem: yes.



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Number Theory From Observations (contd.)

$24 = 2^3 \times 3$ has four prime factors; $30 = 2 \times 3 \times 5$ has three factors; $\Omega(n)$: number of prime factors of n, with $\Omega(1) = 0$.

Pólya's conjecture: $\sum_{k=1}^{n} (-1)^{\Omega(k)} \leq 0.$

- The number of naturals less than n and having an even number of prime factors is at most the number of naturals less than n and having an odd number of prime factors.
- Holds up to $n = 906150257$.

- Disproved by Haselgrove (1958). Showed that the conjecture has a counterexample, and estimated it to be around 1.845×10^{361}.
- The smallest counterexample is $n = 906150257$ (Tanaka 1980).



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Observation and Proof

- We can only observe a finite number of cases.
- The possibilities may be infinite,
  or, even if finite, it can be very large.
- From observation, one makes a conjecture.
- A proof gives the assurance that the conjecture holds for *all* possibilities.
  - Even for possibilities that we are unable to directly verify.



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Euclidean Geometry

Axioms:

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any centre and radius.
4. That all right angles are equal to one another.
5. Playfair's axiom: through a point not on a given line, exactly one parallel line can be drawn which is parallel to the given line.

Using these axioms, Euclid could "prove" all observations about geometry!



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Deduction as Proof

- Given a bunch of statements, we deduce a “new” statement.

A or B; not A; therefore B.	A and B; therefore A.	A implies B; A; therefore B.

- We accept the above arguments as valid/correct/proper.
- But

  A implies B;
B;
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  is not accepted as a valid argument.
- The subject which pertains to the study of arguments is called logic.



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Divorcing Content from Logic

Aristotle introduced syllogisms as valid forms of arguments.
Example:

All men are mortal;
Socrates is a man;
therefore Socrates is mortal.

All horses lay eggs;
Black Beauty is a horse;
therefore, Black Beauty lays eggs.

- The correctness of an argument is in the “structure” of the statements as opposed to the contents of the statements.



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Mathematical Proofs

- Mathematics proceeds from definitions to theorems.
  - Definitions pin down key concepts.
  - Theorems state relations among these concepts.
  - Logical arguments are used to obtain new statements from definitions, axioms and earlier proved theorems.
- The above is a *very* rough description.
  Like saying birds fly by fluttering wings.
- Mathematics is a rich creative process.
  - Pinning down key concepts.
  - Observation, abstraction and boldness in formulating statements.
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Modern Logic: Syntax

- A vocabulary to express statements.
  - Logical symbols (such as $\vee$, $\wedge$, $\neg$, $\forall$, $\exists$), variables.
  - Symbols particular to the subject which is to be reasoned about (function and predicate symbols).
  - Terms built from these function symbols.
  - Statements (atomic formulae and formulae) built from terms using logical connectives.
- Rules of inference: from a set of statements (premisses) infer another (conclusion).
- Theory: a bunch of statements (axioms) in the logic.
- Proof: a sequence of statements, each of which is either a statement in the theory or is derived from previous statements using some rule of inference.
- A theorem in a theory: a statement which can be inferred from the theory using the rules of inference.
- Consistent theory: if there is no statement such that both it and its negation are theorems.



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Modern Logic: Semantics

- Structure: interpretation of constants, functions, predicates.
- Statements (atomic formulae and formulae) evaluate to either true or false.
  - Interpretation of the logical connectives $\wedge$, $\vee$, $\neg$.
  - Intrepretation of the quantifiers $\forall$ and $\exists$.
- Model for a theory: a structure such that all statements of the theory evaluate to true.
- Logical consequence of a theory: if the statement is true in all models of the theory.
- A statement is said to be valid in a theory if it is a logical consequence of the theory.



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Requirement from a Logic

- Soundness: every theorem that can be deduced from a theory is valid.
  - Valid rules of inferences:
    if in a model all the premisses of a rule evaluate to true, then in that model the conclusion should also evaluate to true.
  - A theorem is obtained from the theory by one or more applications of same or different rules of inferences.
- Completeness: every valid statement is a theorem.
  - This shows that the rules of inferences are good enough to deduce in a formal (mechanistic) manner all consequences of the theory.
  - Compared to soundness, usually this is more difficult to show.



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- Developed the first formal system of logic (predicate calculus).
- Analysis of quantified statements and the formal notion of proof.
- Use of logic to reduce some theoretical mathematical statements to simpler logical and mathematical notions.
  - Formal definition of predecessor relation and axioms of number theory.
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- Development of set theory which has become the basic vocabulary for expressing all formal concepts including mathematical and logical notions.
- Formal notion and structure of infinite.
  - A set is infinite if it can be put in 1-1 correspondence with one of its proper subsets.
  - Reals are ‘more numerous’ than naturals.
  - Introduction of cardinal and ordinal numbers.



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- Emphasis on formalism and rigour.
- A founder of proof theory, mathematical logic and metamathematics.
- Hilbert's program asked for the following.
  - All of mathematics follow from a finite system of axioms;
  - Such a system of axioms is provably consistent.
- Entscheidungsproblem.
  - To design an algorithm that will take as input a description of a formal system and a statement in the system and produce as output either "true" or "false" according as whether the statement is true or false.



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Logic and Mathematics (contd.)

Bertrand Russell (1872–1970):

- Russell's paradox: $\mathcal{A} = \{S : S \notin S\}$.
  Does $\mathcal{A} \in \mathcal{A}$?

  If in a small town, the barber shaves every person who does not shave himself, then who shaves the barber?

  - The above paradox struck at the foundation of the scheme to base mathematics on set theory.
  - There are other known examples of self-referential statements.
- Russell-Whitehead:
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